Algebra Qualifying Exam II (May 2023)

You have 120 minutes to complete this exam.

1. (10 points) Let A be an integral domain and let M be a finitely generated torsion module over A. Prove that there is a nonzero element $a \in A$ such that $am = 0$ for all $m \in M$.

2. (10 points) Give an example of three modules A, B and C over a principal ideal domain (PID) such that the sequence
$$0 \to A \to B \to C \to 0$$
is exact, but B is not isomorphic to $A \oplus C$.

3. (10 points) Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ is an injective abelian group.

4. (10 points) Note that the set of 2×2 upper triangular matrices
$$A := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{C} \right\}$$
forms a ring under the usual matrix addition and multiplication. Consider the subsets
$$M_1 := \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \;\middle|\; a \in \mathbb{C} \right\},$$
$$M_2 := \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \;\middle|\; b, c \in \mathbb{C} \right\}.$$
   (a) Prove that M_1 and M_2 are both left ideals of A.
   (b) Prove that M_1 and M_2 are both projective left modules over A.

5. (10 points) Compute the group
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/12, \mathbb{Z}/18).$$

6. (10 points) Recall the principal ideal domain of Gaussian integers
$$\mathbb{Z}[\sqrt{-1}] := \left\{ a + b\sqrt{-1} \;\middle|\; a, b \in \mathbb{Z} \right\}.$$
Let $J \in \mathrm{GL}_n(\mathbb{Z})$ be an integer matrix such that
$$J^2 = -I_n,$$
where I_n is the identical matrix. We view $\mathbb{Z}^n$ as a module over $\mathbb{Z}[\sqrt{-1}]$ such that
$$\forall a + b\sqrt{-1} \in \mathbb{Z}[\sqrt{-1}],\ \forall \vec{v} \in \mathbb{Z}^n, \qquad (a + b\sqrt{-1}) \cdot \vec{v} := a\vec{v} + bJ\vec{v}.$$
   (a) Prove that $\mathbb{Z}^n$ is a free module over $\mathbb{Z}[\sqrt{-1}]$.
   (b) Determine the rank of $\mathbb{Z}^n$ as a module over $\mathbb{Z}[\sqrt{-1}]$.